Quantum Cascade Laser Figure of Merit Derivation

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1 Figure of merit the easy way

The figure of merit (FoM) for a quantum cascade (QC) laser (QCL) is a tool used to maximize performance from a QC design. The FoM represents the parameters from the QC laser gain coefficient that are affected by the quantum design of a QC structure. The QCL gain coefficient is generally given as

$$g_c = \tau_u \big(1 - \frac{\tau_\ell}{\tau_{u\ell}}\big) \frac{4\pi q z_{u\ell}^2}{\lambda_0 \epsilon_0 n_{eff} L_p} \frac{1}{\delta\varepsilon_{u\ell}} \;\; [\tfrac{\text{length}}{\text{current}}] \tag{1}$$

where τ_u is the lifetime of the upper laser state, τ_ℓ is the lifetime of the lower laser state, $\tau_{u\ell}$ is the transition time between the upper and lower laser state, q is the electron charge, $z_{u\ell}$ is the optical dipole matrix element (having units of length), λ_0 is the free space wavelength of the lasing transition, n_{eff} is the effective refractive index of the optical mode, L_p is the length of a single active-injector region period, and $\delta\varepsilon_{u\ell}$ is the full-width-at-half-maximum (FWHM) of transition's spontaneous emission (usually taken as the measured FWHM of a QC structure's electroluminescence, and in units of energy).

Since the optical dipole matrix element $z_{u\ell}$ and the energy state lifetimes τ_i are QC design-dependent parameters, the most direct approach to deriving a FoM is simply to pull these factors from the gain coefficient.

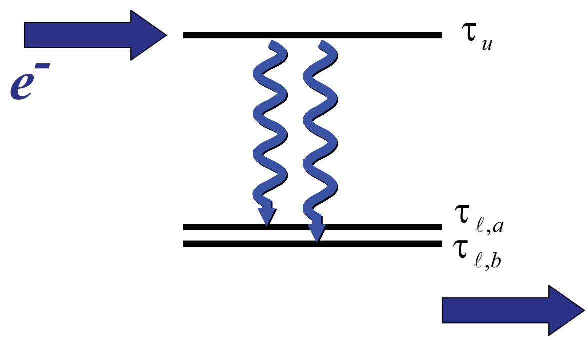
$$\text{FoM} = \tau_u \big(1 - \frac{\tau_\ell}{\tau_{u\ell}}\big) z_{u\ell}^2 \;\; [\text{time} \times \text{length}^2] \tag{2}$$

While this is the simplest method for deriving a FoM, it has severe limitations when practically implemented. QC lasers after all, comprise a system of coupled quantum wells with significant intermixing (anticrossing) of quantum states; in Eq. (2), we have assumed only a single upper laser state and a single lower laser state.

2 Intermixed lower energy states

For a proper calculation of the laser gain coefficient—and thus FoM—we must include all optical transitions that might contribute photons into the lasing mode. One might address this problem by perturbing the field at which the FoM parameters are calculated, so as to prevent the upper and lower laser states from intermixing with nearby states. However, the FoM itself is a field-dependent parameter, so this approach is not ideal for calculating laser gain at current turn-on (i.e., the design field).

To more broadly address the problem of multiple charge carrier transitions contributing photons to the lasing mode, we need to consider individually each potential transition's contribution to the optical gain. As an example, let's examine an optical transition with one upper energy state and two closely spaced, intermixed lower energy states.



We need to remember that each transition has its own spontaneous emission spectrum with a finite FWHM; in intersubband heterostructure emitters, a good estimate for $\delta\varepsilon_{u\ell}$ is 10% of the transition energy. If the two transitions are close enough together, the spontaneous emission will overlap in energy. Let's assume the spontaneous emission lineshape $\mathcal{L}$ for a transition with energy $\varepsilon_{u\ell}$ has a Lorentzian form.

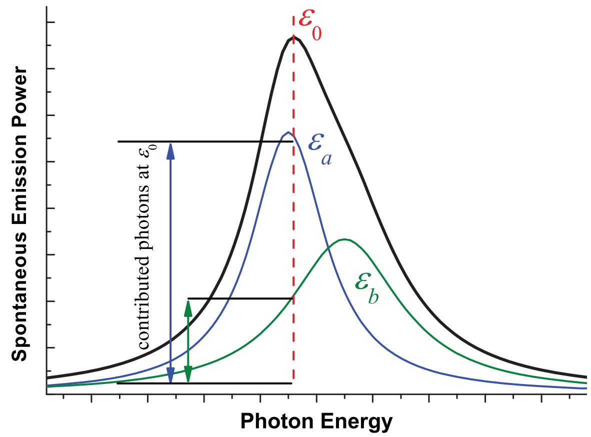
$$\mathcal{L}(\varepsilon_{u\ell}) = \frac{1}{(\frac{\varepsilon_{u\ell}-\varepsilon_0}{\delta\varepsilon_{u\ell}})^2 + 1} \tag{3}$$

Here, ε_0 is the lasing photon energy. Now, each transition will be able to contribute photons into the lasing mode. The lasing wavelength will not be exactly that of the dominant optical transition, but somewhere between the two transitions. To find the lasing wavelength, we need to keep in mind that the transitions might have different spontaneous emission rates. That is, one transition might be able to emit photons faster than the other. The spontaneous emission rate W_{sp} $[\frac{1}{\text{time}}]$ is the inverse of the spontaneous emission lifetime τ_{sp} [time].

$$W_{sp} = \frac{1}{\tau_{sp}} = \frac{8\pi^2 q^2}{\hbar\epsilon\lambda^3} z_{u\ell}^2 \ \ [\tfrac{1}{\text{time}}] \tag{4}$$

Notice that the rate, or the strength of the transition, is proportional to $z_{u\ell}^2$. If we add up the spontaneous emission spectrum of each transition multiplied by $z_{u\ell}^2$, the peak gain and thus the lasing energy ε_0 can be found.

$$\begin{aligned}\varepsilon_0 &= max\Big(\mathcal{L}(\varepsilon_{u\ell,a})z_{u\ell,a}^2 + \mathcal{L}(\varepsilon_{u\ell,b})z_{u\ell,b}^2\Big)\\ &= max\Big(\sum_{\ell}\mathcal{L}(\varepsilon_{u\ell})z_{u\ell}^2\Big) \quad \text{[energy]} \end{aligned} \tag{5}$$



Now that we know what the lasing energy is going to be, let's look at stimulated emission for each of the transitions. Gain γ is simply the charge carrier population difference $N = N_u - N_\ell$ $[\frac{1}{\text{volume}}]$ multiplied by the transition cross-section σ [area].

$$\gamma(\varepsilon_{u\ell}) = N \times \sigma(\varepsilon_{u\ell}) \ \ [\tfrac{1}{\text{length}}] \tag{6}$$

Let's take each of the two contributions to γ individually. In a simple two level optical transition system with an upper state pumped at rate R_u $[\frac{1}{\text{volume}\times\text{time}}]$, the carrier population difference is

$$N = R_u\tau_u\big(1 - \frac{\tau_\ell}{\tau_{u\ell}}\big) = \frac{J}{q}\frac{1}{N_pL_p} \ \ [\tfrac{1}{\text{volume}}] \tag{7}$$

where J is pumping current density $[\frac{\text{current}}{\text{area}}]$ and N_p is the number of active region-injector periods in the QCL active core. Note that N represents the total population inversion from the set of *all* active regions in the active core. In this way, N describes the active core gain region as a whole. If one wishes to consider only population inversion for a single QC active region Δn, the value is usually given in terms of sheet density.

$$\Delta n = N\frac{L_p}{N_p} \ \ [\tfrac{1}{\text{area}}] \tag{8}$$

The transition cross-section can be found by recognizing that, at threshold where optical gain clamps,

$$\gamma(\varepsilon_{u\ell}) = \alpha_{total} = g_c \Gamma J_{th} \ \ [\tfrac{1}{\text{length}}] \tag{9}$$

where α_{total} is the total optical loss (waveguide and mirror loss), Γ is the gain region confinement factor for the optical mode in the waveguide, and J_{th} is the threshold current density $[\frac{\text{current}}{\text{area}}]$. By applying Eqs. (1), (6) and (7) to Eq. (9), we get

$$\sigma(\varepsilon_{u\ell}) = \frac{g_c \Gamma J_{th}}{N} = g_c \Gamma \frac{q N_p L_p}{\tau_u \big(1 - \frac{\tau_\ell}{\tau_{u\ell}}\big)} = \frac{\Gamma q^2 4\pi N_p z_{u\ell}^2}{\lambda_0 \epsilon_0 n_{eff}\ \delta\varepsilon_{u\ell}} \ \ \text{[area]}. \tag{10}$$

While this is close for the transition cross-section, we've got one correction to make. The gain coefficient from Eq. (1) was derived assuming two discreet states. To correct for this, we need to multiply the original g_c by our lineshape function $\mathcal{L}(\varepsilon_{u\ell})$ from Eq. (3). Thus, we get the quantum cascade laser transition cross-section.

$$\sigma(\varepsilon_{u\ell}) = \frac{\Gamma q^2 4\pi N_p z_{u\ell}^2}{\lambda_0 \epsilon_0 n_{eff}\ \delta\varepsilon_{u\ell}} \mathcal{L}(\varepsilon_{u\ell}) \ \ \text{[area]} \tag{11}$$

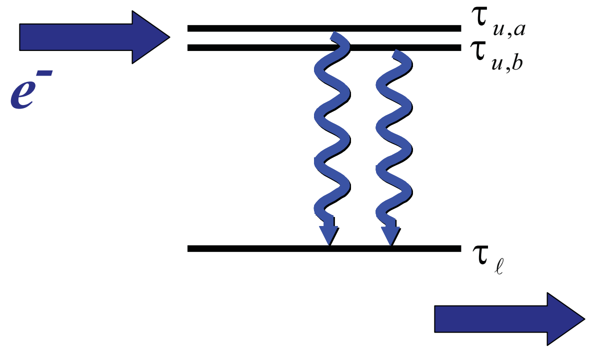
Note, again, this quantity describes the active core gain region as a whole.

Now, to find the FoM for our example with one upper energy state and two closely spaced lower energy states, we can pull from the components of Eq. (6) those elements that are influenced by quantum design.

$$\begin{aligned} \text{FoM} &= \tau_u \big(1 - \frac{\tau_{\ell,a}}{\tau_{u\ell,a}}\big) z_{u\ell,a}^2 \mathcal{L}(\varepsilon_{u\ell,a}) + \tau_u \big(1 - \frac{\tau_{\ell,b}}{\tau_{u\ell,b}}\big) z_{u\ell,b}^2 \mathcal{L}(\varepsilon_{u\ell,b}) \\ &= \sum_\ell \tau_u \big(1 - \frac{\tau_\ell}{\tau_{u\ell}}\big) z_{u\ell}^2 \mathcal{L}(\varepsilon_{u\ell}) \ \ [\text{time}\times\text{length}^2] \end{aligned} \tag{12}$$

3 Intermixed upper energy states

Calculating a proper FoM for energy transitions with upper state intermixing takes consideration similar to the lower state intermixing case. However, with upper state intermixing, we have an additional complication. We cannot assume that each upper energy state is equally populated with electrons; the relative populations of each upper state will influence the total gain contributed by that transition.



Recall that our basic relation for gain,

$$\gamma(\varepsilon_{u\ell}) = R_u \tau_u \big(1 - \frac{\tau_\ell}{\tau_{u\ell}}\big) \sigma(\varepsilon_{u\ell}) \;\; [\tfrac{1}{\text{length}}] \tag{13}$$

has a term R_u that describes the rate at which the upper energy level u is populated with electrons. For a FoM calculation, where we are focused on obtaining a number that allows us to compare different QC designs, we are not concerned about the absolute pumping rate. Rather, what we need is a weighting factor that reflects the relative population of each of the upper energy states. Now, we can write our FoM as

$$\text{FoM} = \sum_u C_u \tau_u \big(1 - \frac{\tau_\ell}{\tau_{u\ell}}\big) z_{u\ell}^2 \mathcal{L}(\varepsilon_{u\ell}) \;\; [\text{time}\times\text{length}^2] \tag{14}$$

with an upper state weighting factor C_u. As we've previously shown, the pumping rate R_u of an energy state is proportional to the current density passing through the state. In a QCL system with strong coupling between energy states,

$$J_u \approx \frac{q n_u}{2\tau_u} \;\; [\tfrac{\text{current}}{\text{area}}] \tag{15}$$

where n_u is the sheet density $[\frac{1}{\text{area}}]$ of electrons populating the state. Thus, our weighting factor $C_u \propto n_u/\tau_u$. The energy state population n_u follows the Fermi-Dirac distribution

$$n_u = n_s \frac{1}{e^{-\frac{\Delta\varepsilon}{k_B T}} + 1} \;\; [\tfrac{1}{\text{area}}] \tag{16}$$

where n_s is the sheet density of electrons in the injector, $\Delta\varepsilon = \varepsilon_u - \varepsilon_F$ is approximately the energy difference between the state u and the injector ground state, k_B is the Boltzmann constant, and T is temperature. We can ignore including a factor for density of states if we assume low injector doping densities—generally the case for QC lasers—and a reasonable operating temperature. Also under these conditions, the Fermi-Dirac distribution is well-approximated by the Boltzmann distribution, so

$$n_u \propto e^{-\frac{\Delta\varepsilon}{k_B T}} \tag{17}$$

and

$$C_u \propto \frac{e^{-\frac{\Delta\varepsilon}{k_B T}}}{\tau_u} \tag{18}$$

Because C_u is a weighting factor whose sum over all states represents a total electron flux, the restriction $\sum_u C_u = 1$ must hold.

$$C_u = \frac{\frac{e^{-\frac{\Delta\varepsilon}{k_B T}}}{\tau_u}}{\sum_u \frac{e^{-\frac{\Delta\varepsilon}{k_B T}}}{\tau_u}} \tag{19}$$

Finally, we arrive at the general equation for FoM by combining Eqs. (12) and (14), the generalized cases for transitions between intermixed upper and lower energy states.

$$\mathrm{FoM} = \sum_{u,\ell} C_u \tau_u \big(1 - \frac{\tau_\ell}{\tau_{u\ell}}\big) z_{u\ell}^2 \mathcal{L}(\varepsilon_{u\ell}) \;\; \text{[time}\times\text{length}^2\text{]} \tag{20}$$